Basics of
CALCULUS
Simplified

A. A. Tilak. B.E. (Electrical)
Some guidelines for reading this book:

- All the chapters in this book are arranged in a logical order, so that readers can read them sequentially.
- The figures and graphs are placed on the same page or the facing page where their description appears. This eliminates the need for numbering the figures and connecting them to the texts.
- Readers may find repetition of some ‘text material’ in different chapters. This was seen as a better option than to ask readers to go back and forth whenever a reference is made to a text from a different chapter. Repetition has also been used to highlight and drive home the importance of a particular ‘text’.
- Study of calculus should not be about memorizing differentiation and integration formulae. The main focus of the readers should be on understanding and appreciating the fundamental logic and basic concepts of calculus.
- This is not intended to be a textbook of calculus. The main objective of this book is to clearly explain basic concepts of calculus. Accordingly more emphasis was given to include different interpretations of basic ideas of calculus, rather than incorporating proofs of all the theorems of calculus. Hence the readers may find that some related sub-topics of few subjects are not included in this book or some topics have not been covered in as much detail.

- **Conceptual understanding** and **analytical abilities** are the two most important qualities essential to attain a certain level of proficiency in any subject. This is however particularly important in case of a subject like calculus. **Conceptual understanding** of calculus basics can be developed by reading theory books like the one you are about to read. However, **Analytical abilities** cannot be developed by merely reading books. **The only way to develop analytical skills in calculus is to practice solving as many different calculus problems as possible from text books or from any other source.**

- Readers are welcome to send their suggestions, comments and feedback about this book to following email address.
  
  *mail@targetpublications.org*
PREFACE

Students passing out of school and entering Junior College approach the subject of Calculus with a great deal of nervousness, if not an outright sense of dread. The subject is looked upon more as a challenge to somehow overcome in college than gaining knowledge of what may be described as the crown jewel of mathematics. Advanced text books on the subject tend to get overly technical, reinforcing the adverse impression among the beginners that Calculus is a tough subject, and the smart way is to somehow ‘clear’ the subject by going over it mechanically with the help of guides.

I have carried within me this disturbing feeling for long, that instead of being seen as a fascinating subject that should fire students’ imagination, Calculus is just seen as a troublesome subject.

This book on the basics of Calculus, if this presentation may be so described, is thus a humble attempt to introduce Calculus to youngsters just out of school, in a language with which they would feel at ease, helping them with liberal use of examples drawn from our day to day life. All the topics are explained in a structured manner so that students do not miss out on the fundamental logic and basic concepts of Calculus. The physical, graphical, algebraic and geometric interpretations of basic ideas of Calculus are explained in such great detail that students can look at Calculus from a refreshing viewpoint and dispel the impression that it is a tough subject.

The basic approach followed here for getting the subject across was to let it unfold itself as a flowing narrative. Yet care has been taken to ensure that no part of the contents would suffer from errors in the exposition of the basics.

From ‘thinking’ to the ‘finished draft’ of the book was a long and difficult journey, which was only made possible with the help from many friends. I am extremely happy to express my gratitude to all of them.

Prof. Dr. Sharad S. Sane from IIT-Bombay checked the initial draft of the first few chapters and gave his word of encouragement to undertake this project. He also checked the final draft and gave several important suggestions. Prof. Dr. Vasant M. Kane, former H.O.D. Mathematics and Vice Principal of RKT College, Prof. Prakash G. Dixit, H.O.D. Statistics, Modern College, Prof. C.V. Rayarikar, former H.O.D. Mathematics, Modern College, Prof. Mrs. S. A. Joshi from Ruia College and Mr. A.B. More extended all the help in editing the book and gave extremely useful and valuable comments for improving the text.

Mr. A.L. Narasimhan, a colleague and a friend for several years, extended his help in editing several drafts at various stages. His facility with language has greatly contributed to the flowing text that the readers may notice.

My acknowledgement of gratitude may not be complete without mentioning my friends Mr. J. V. Kadhe and Mr. V. A. Patwardhan, who spared no efforts to lead me to valuable contacts, which were of great help in editing the book.

Lastly I would like to submit to such of my readers who are learned scholars of mathematics that I am acutely aware that this book, both in its contents and style, will seem to them to be rather elementary and incomplete, and will lend itself to much improvement. I seek their indulgence and wish to stress again that, this book is essentially addressed to beginners in their study of Calculus, and if it succeeds in interesting them and making them seek closer study through more comprehensive books, I shall consider my labour amply rewarded.

A. A. Tilak
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Introduction

Geometry is the study of shapes.
Algebra is the study of operations and their application to solving equations.
Calculus is the mathematical study of motion and change.

Dictionary defines calculus as “a particular method of mathematical calculation or reasoning”. While being literally correct, this definition grossly understates the sweeping influence that this branch of mathematics has had on the scientific and technological advances that have come about in the last 300 years.

Great thinkers from ancient times were fascinated by giant planets of our solar system and countless stars from galaxies, which were all in a state of perpetual motion or in a state of constant change in vast space and time. Life on earth reflected this cosmic reality as everything in it including the living beings was characterized by movement and change, be it the movement of rain clouds across the continents, sprinting athletes, galloping horses, flowing rivers or change in seasons.

Thus, any sense of control over, or intelligent coexistence with this dynamic world would rest on man’s ability to clearly understand this phenomenon of movement and change and acquire the knowledge to predict its behavior. The search for a universal order behind all motion and change started as mere philosophical contemplation, but evolved over several centuries of research by many distinguished mathematicians, until Isaac Newton, from England and Gottfried Leibniz, from Germany, independently developed a ‘new mathematical method’ around the year 1665, which explained with precision and clarity, the phenomenon of motion and change.

This ‘new mathematical method’ is what we all know as ‘Calculus’.

We may trace calculus to its earliest roots with the 5th century B.C. Greek philosopher Zeno, who postulated his famous paradoxes about motion. These paradoxes were the earliest seeds of the concepts of calculus. Zeno questioned the universal observation that when an arrow is shot from a bow, it starts moving rapidly through the air and ultimately hits its target. He argued that “the fast moving arrow is however stationary in one place at any instant of time. How could one say then that the arrow is also moving at those instants of time?” He surmised theoretically that the arrow can never hit its target as it is stationary at any instant of time.

But it is also true that the arrow does move through the air very rapidly and reaches its target. That the arrow has a certain speed or velocity at all the instants where it appears to be stationary is a concept developed in Calculus. Speed or velocity was the only term used earlier to completely describe all the aspects of motion. However, a new concept of the ‘Instantaneous velocity’ was developed to describe the ‘rate of change’ in motion. Calculus thus resolved Zeno’s paradoxes after nearly two millenniums.

The entire concept of Calculus is based upon understanding of its two main ideas, Derivative and Integral. These fundamental ideas of Calculus can be understood by observing and analyzing common day-to-day examples, as explained in the following chapters. Calculus is all around us. When we are driving a car and see how fast we are moving, there is Calculus. When we hit a cricket ball and see where it lands, there is Calculus. When we observe planets moving around the Sun, there is Calculus. And when
we launch a satellite into the orbit, there is Calculus.

Calculus has been extremely effective as it not only enables us to understand our ever changing world, but also enables us to predict the changes and control them. We can calculate where the cricket ball will land when we hit it, or launch a satellite with predetermined speed to place it into a geostationary orbit.

Calculus gave mankind a scientific tool that perhaps has been most influential in setting off the spectacular advances in science and technology, from physics to engineering, chemistry to biology and from business to economics and in many other fields.

In fact, it can be said that ‘Calculus’ is indeed one of the greatest intellectual triumphs of humanity.

Note :- Calculus makes extensive use of basic mathematics. Hence a revision of basic mathematics will help a reader to understand the concepts of calculus better.

With this in mind, a brief review of a few related topics which are frequently used in study of calculus is covered in the first chapter of ‘Pre-calculus review’.
Let us briefly review a few topics of basic mathematics like Graphs, Algebraic equations, Functions and Trigonometric functions which are extensively used in the study of calculus. (Readers who are confident of their grounding in these topics can directly move on to the next chapter on Derivative).

1.1 Interpretation of equations & graphs

Let us take a simple equation like \( y = x^2 - 3 \) and see how to plot a graph for this equation. Let us assign successive values of \( x \) in the equation and find out corresponding values of \( y \) and tabulate them as given below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>6</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
<td>6</td>
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We can draw these points in \( X \ Y \) coordinates system to plot a graph to represent the equation. This method of plotting a graph is called as ‘Point plotting’.

We can observe that, it is a parabola graph.

1.2 Intersection of graph with \( X \) or \( Y \) axis

The important information that we need to know about the graph is as to where it crosses the \( X \) axis or the \( Y \) axis.

The graph of an equation crosses the \( X \) axis when value of \( y \) is zero. We can substitute \( y = 0 \) in the equation to find out where it crosses the \( X \) axis.

\[
\begin{align*}
  y &= x^2 - 3 \\
  \therefore 0 &= x^2 - 3 \\
  \therefore x^2 &= 3 \\
  \therefore x &= \pm \sqrt{3}
\end{align*}
\]

We can see that the graph crosses the \( X \) axis at two points, \( x = \sqrt{3} \) and \( x = -\sqrt{3} \).

Let us get the \( Y \) axis intersect of the equation by substituting \( x = 0 \) in the equation. We will get \( y = -3 \) when we substitute \( x = 0 \). Hence the graph crosses the \( Y \) axis at \( -3 \).

1.3 Symmetry of a graph

The other important information about the graph is to understand its symmetry.

A graph is said to be symmetrical about the \( Y \) axis when point \((x, y)\) and its mirror image \((-x, y)\) are both located on the graph.

Similarly, the graph is symmetrical about \( X \) axis if point \((x, y)\) and point \((x, -y)\) are both located on the graph.
There are graphs which are symmetrical around the origin. In this case both the points \((x, y)\) and \((-x, -y)\) are located on the graph. These two points are also located on a line that passes through the origin as seen below.

We have got back to the same equation. In other words, point \((x, y)\) and \((-x, -y)\) are both there on this graph. Hence this equation is symmetrical around the origin.

The graph for the equation \(y = x^3 - 9x\) is given below, which confirms this symmetry.

In general, any polynomial equation which is a sum of terms that involve only odd powers of \(x\) is an odd function of \(x\), and hence is symmetrical about the origin.

The symmetry of equation about either \(X\) or \(Y\) axis can also be checked by substitution method.

### 1.4 Example

We have an equation \(y = x^3 - 9x\). The objective is to find out whether the graph of this equation is symmetrical around the origin without plotting it.

Let us find out algebraically whether the graph of this equation is symmetrical around the origin by substituting values of \(x\) with \(-x\) and \(y\) with \(-y\) in the equation and check whether the resulting equation remains same.

\[
\begin{align*}
  y &= x^3 - 9x \\
  \therefore \quad -y &= (-x)^3 - 9(-x) \\
  \therefore \quad -y &= -x^3 + 9x \\
  \therefore \quad y &= x^3 - 9x
\end{align*}
\]

1.5 Intersection of parabola & line

Let us consider two equations and find out the points where their graphs intersect each other.

\[x^2 - y = 3\] and \(x + y + 1 = 0\).

After rearranging and simplifying these equations, we can get following equations which are of a similar format.

\[y = x^2 - 3\] and \(y = -x - 1\).

The first equation is of a parabola and the second one is of a straight line.
Now let us equate these two equations to find out their points of intersections.

\[ x^2 - 3 = -x - 1 \]
\[ x^2 + x - 2 = 0 \]
\[ x(x + 2) - 1(x + 2) = 0 \]
\[ (x + 2)(x - 1) = 0 \]

Hence \( x = -2 \) or \( x = 1 \)

Let us substitute these two values of \( x \) in any one of the above equations and get coordinates of two points of intersections, \((-2, 1)\) and \((1, -2)\).

The earlier figure shows two points of intersections of these two equations represented by parabola and line graphs.

### 1.6 Slope of a line

Slope of a non-vertical line passing through two points \((x_1, y_1)\) and \((x_2, y_2)\) is given as follows.

\[
\text{Slope } m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

### 1.7 Equation of a line

Let us find the slope and the equation of a line passing through two points \((1, -2)\) and \((3, 1)\).

Slope \( m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta Y}{\Delta X} = \frac{1 - (-2)}{3 - 1} = \frac{3}{2} \)

Let us make use of this slope formula and use slope quotient and coordinates of point \((3, 1)\) to create an equation for the line.

\[
\text{Slope } m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3}{2}
\]

\[
\therefore \frac{y - 1}{x - 3} = \frac{3}{2}
\]

By rearranging, let us get the following ‘Point slope form’ of line equation.

\[
\therefore y - 1 = \frac{3}{2}(x - 3)
\]
The same equation can be simplified to another form as given below.

\[
y = \frac{3}{2}x - \frac{7}{2}
\]

This is known as the ‘Slope intercept form’ of line equation. The number before \( x \) in this line equation indicates the slope of the line, which is \( \frac{3}{2} \) in our example. This line equation also indicates that, this line intersects the \( Y \) axis at point \( -\frac{7}{2} \).

(When we substitute \( x = 0 \) in the slope intercept form of equation, we will get \( y = -\frac{7}{2} \) which is the point of intersection of the line with the \( Y \) axis).

Let us eliminate the denominator in the ‘Point slope form’ of equation by multiplying both the sides of the equation by 2.

\[
\begin{align*}
y - 1 &= \frac{3}{2}(x - 3) \\
2(y - 1) &= 2 \times \frac{3}{2} (x - 3) \\
2y - 3x + 7 &= 0
\end{align*}
\]

This equation is called as the ‘General form’ of line equation.

The graph for this equation \( y = \frac{3}{2}x - \frac{7}{2} \) is shown as a line passing through two points \((1, -2)\) and \((3, 1)\). It is also showing its point of intersection to the \( Y \) axis at \((0, -\frac{7}{2})\).

\textbf{Note} :- Equation of a line of given slope ‘\( m \)’ and passing through point \((x_1, y_1)\) in ‘Point slope form’ is \( y - y_1 = m (x - x_1) \).

(The slope is zero for a horizontal line. The slope is not defined for a vertical line.)

\[ \textbf{1.8 Parallel lines} \]

Let us find an equation for a line that passes through point \((2, 1)\) and is parallel to the earlier line \(2y - 3x + 7 = 0\).

Let us rewrite this equation in ‘Slope intercept form’ as follows.

\[
y = \frac{3}{2}x - \frac{7}{2}
\]

The parallel line passing through point \((2, 1)\) will have the same slope as the slope of the earlier line which is equal to \( \frac{3}{2} \).

Let us use ‘point slope formula’ to write the equation for a line passing through point \((2, 1)\) and having a slope of \( \frac{3}{2} \) as follows.

\[
y - (1) = \frac{3}{2} (x - 2)
\]

Let us simplify this equation and rewrite it in ‘Slope intercept form’ as given below.

\[
y = \frac{3}{2}x - 2
\]

Let us further simplify and write it in a ‘General form’ as follows.

\[ 2y - 3x + 4 = 0 \]

The slope of this new line is \( \frac{3}{2} \), and it passes through new point \((2, 1)\) as required. We can check it by substituting coordinates of this point in place of \( x \) and \( y \) in the equation. Let us also observe that this new line intersects the \( Y \) axis at point \( y = -2 \).
1.9 Perpendicular lines

Let us find out the equation of a line which passes through point \((1, -2)\) and is perpendicular to the line \(2y - 3x + 7 = 0\).

The slope of a line perpendicular to a given line has a slope which is the negative reciprocal of the slope of the line to which it is perpendicular.

Hence the slope of the perpendicular line would be \((-\frac{1}{3/2})\), i.e. \(-\frac{2}{3}\).

Let us write equation for a perpendicular line by using ‘Point slope formula’.

\[ y - (-2) = \frac{2}{3}(x - 1) \]

Let us simplify this equation and write it in ‘Slope intercept form’ as given below.

\[ y = \frac{2}{3}x - \frac{4}{3} \]

Let us further simplify and write it in a ‘General form’ as follows.

\[ 3y + 2x + 4 = 0 \]

Let us list some conclusions from the study of equations of horizontal and perpendicular lines.

- Slope intercept form of line equation is \(y = mx + b\) where \(m\) is the slope and \(b\) is the \(Y\) axis intersection point.
- Lines horizontal to the \(X\) axis have equation \(y = b\).
- Lines perpendicular to the \(X\) axis have slopes equal to ‘infinity’. Hence their slopes are not defined. However their equation would be \(x = c\).

1.10 Centigrade to Fahrenheit conversion

Let us evolve an equation to convert temperature from Centigrade to Fahrenheit based on following information.

- Water freezes at \(0^\circ\) C or at \(32^\circ\) F.
- Water boils at \(100^\circ\) C or at \(212^\circ\) F.

Let us identify two points, \((0, 32)\) and \((100, 212)\) as the two points connecting a line. Let us first find the slope of a line connecting these two points as follows.

\[ \text{Slope } m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5} \]

Let us select this slope and one point \((0, 32)\) and write the equation using ‘Point slope formula’ as follows.

\[ F - 32 = \frac{9}{5} (C - 0) \]

\[ \therefore F = \frac{9}{5} C + 32 \]

This equation is used to convert temp. from Centigrade to Fahrenheit and vice versa.

1.11 Functions

There are many relationships where the value of the ‘output variable’ depends on the varying value of the ‘input variable’. For example, the area of a circle depends on its radius, the total distance travelled by a car in an hour depends on its speed, the time to fill a water tank depends on the speed of water flow, the monthly sales revenue of a product depends on the quantity of product sold etc.

All such relationships can be expressed by suitable equations. The area of a circle can be expressed by the equation \(A = \pi r^2\). Here, the value of the area of a circle \(A\) depends on the value of its radius \(r\). Hence, \(A\) is the ‘dependent output variable’ and \(r\) is the ‘independent input variable’. This equation gives a unique value of area \(A\) for any given value of radius \(r\). Hence this equation is a ‘Function’.

The equation can be called a function only if it gives a unique value of the ‘dependent output variable’ for any value of the ‘independent input variable’.

Let us draw a graph for \(y = x^3 - 9x\) and check whether this equation is a function.
This equation would give us different unique values of ‘output variable \( y \)' if we substitute different values of ‘input variable \( x \)' in this equation. Hence this equation can be described as a function.

But all equations are not functions. Let us check equation \( x = y^2 - 4 \) and see why it does not qualify to be a function.

First let us check where the graph of this function intercepts the X and Y axes. We get \( x = -4 \) when we substitute \( y = 0 \) in this equation, which means that the graph of this equation intercepts X axis at point \(-4\). We also get \( y = +2 \) and \(-2\) when we substitute \( x = 0 \) in this equation, which means that the graph of this equation intercepts the Y axis at two points, \( y = +2 \) and \(-2\). This equation is not a function because it gives more than one value of \( y \) for one value of \( x \).

The domain of the function is the set of all the values of the ‘independent input variable \( x \)' for which the function is defined. The range of the function is the set of all the values taken on by the ‘dependent output variable \( y \)’.

Let us see a simple rule which can help us to determine if the equation represented by a graph is a function or not a function.

We can draw a vertical line anywhere on the graph and check if it crosses the graph more than once. The ‘Vertical line rule’ states that, the equation is a function only if the vertical line drawn anywhere on its graph crosses the graph only at one point. Let us see some typical functions and their graphs and observe that the vertical line drawn anywhere on these graphs crosses the graph only at one point in spite of their different shapes.

Square root function \( f(x) = \sqrt{x} \) is a function if its domain is \( x > 0 \).

Function \( f(x) = x^2 \) is a function of a parabola. Its graph is given below.
Chapter 01: Pre-Calculus Review

The graph for a function \( f(x) = \frac{1}{x-1} \) is given below.

We can observe that we cannot substitute \( x = 1 \) in this function as it will make the denominator zero. Hence the ‘Domain’ of this function is \( x > 1 \) and \( x < 1 \) or \( x \neq 1 \). We can see from the graph that the value of \( y \) of this function approaches infinity (or negative infinity), as value of \( x \) approaches 1 from the right or the left side. Hence, line \( x = 1 \) is the ‘vertical Asymptote’ of this graph.

The graph of an absolute value function \( f(x) = |x| \) is given below. (Absolute value of any number is always positive or non-negative).

Let us see a function which is defined by more than one rule.

\[
f(x) = \begin{cases} 
1 - x & \text{if } x < 1 \\
\sqrt{x - 1} & \text{if } x \geq 1
\end{cases}
\]

First rule.

Second rule.

The graphical representation of this function is a straight line going through points \((1, 0)\) and \((0, 1)\) for the first rule, and a ‘square root type graph’ shifted by one unit on X axis for the second rule.

1.12 One-to-one function

A function is one-to-one between the ‘set of \( x \) and the set \( y' \) if each value of \( y \) in the ‘Range’ also gives exactly one value of \( x \) in the ‘Domain’.

We have seen that an equation is a function if for each unique value of ‘independent input variable \( x' \), it gives a unique value of ‘dependent output variable \( y' \). However one-to-one function is the one where the reverse property is also true, i.e. for each unique value of \( y \), it also gives a unique value of \( x \).

Function \( f(x) = x^2 \) is not a one-to-one function, because for a given value of \( y = 4 \), we get two values of \( x \), namely \( x = 2 \) and \( -2 \).
Let us see a simple rule which can help us to determine if the equation represented by a graph is a one-to-one function or not.

The equation is a one-to-one function only if the horizontal line drawn anywhere on its graph crosses the graph only at one point. This is called as the ‘Horizontal line rule’.

Function \( f(x) = x^3 \) is a one-to-one function. If we take a look at the graph of this function, then we can notice that the horizontal line drawn anywhere on graph, crosses the graph at only one point. Hence, for any value of \( y \), there is one unique value of \( x \), and vice versa.

We have seen that the ‘Vertical line test’ is used on a graph to determine whether that equation is a function. And a ‘Horizontal line test’ is used on a graph to determine whether that equation is a one-to-one function.

Let us summarize the list of functions and note their characteristics that we have studied so far.

- \( f(x) = x \) is a line function.
- \( f(x) = x^2 \) is a parabola function and is an even function.
- \( f(x) = x^3 \) is an odd function.
- \( f(x) = \sqrt{x} \) is a function if its domain is defined as \( x > 0 \).
- \( f(x) = |x| \) is an absolute function.
- \( f(x) = x + 5 \) is a shift function.
- Even functions are symmetrical around \( Y \) axis. \( f(-x) = f(x) \). e.g. \( f(x) = x^2 \) or \( x^4 \) or \( x^6 \) etc.

- Odd functions are symmetrical around origin. \( f(-x) = -f(x) \). e.g. \( f(x) = x^3 \) or \( x^5 \) or \( x^7 \) etc.

We can analyze and interpret complicated functions by studying these simple functions, as shown in the next example of a shift function.

1.13 Shift functions

Let us analyze the function \( f(x) = \sqrt{x - 3} + 5 \).

First part of this function is \( \sqrt{x - 3} \). Here value of \( x \) cannot be less than 3, as it will give us square root of a negative number or an imaginary number. Hence the ‘Domain’ of this function is \( x \geq 3 \). It is a square root function shifted 3 units to the right on \( X \) axis. But it also has ‘4 5’. Hence the origin of the graph will be shifted by 5 units on \( Y \) axis. Hence the starting point of this graph will be \((3, 5)\) as we can see from the following graph.

1.14 Trigonometric functions

Let us use the unit circle definition for all the trigonometric functions.

Radian measurement system for measuring angles is as follows.

\[ 360^\circ = 2\pi, \quad 180^\circ = \pi, \quad \text{and} \quad 90^\circ = \pi/2. \]
The **sine** function is a wave function which varies between $-1$ and $1$. It is an odd function which is symmetrical around the origin.

The **cosine** function is an even function which is symmetrical around the $Y$ axis.

Tangent function is not defined at $\pi/2$ or $3\pi/2$ or at $-\pi/2$ or $-3\pi/2$. Because at all these values $\cos \theta = 0$. And division of any numerator by zero is not possible, hence the $\tan \theta$ function is not defined at these values of $\theta$.

\[
\sin \theta = y \quad \cos \theta = x \\
cosec \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \\
\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta} \\
\sin^2 \theta + \cos^2 \theta = 1
\]